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THE LAPLACE TRANSFORM OF THE FOURTH MOMENT OF THE ZETA-FUNCTION

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Abstract. The Laplace transform of $|\zeta(\frac{1}{2}+ix)|^4$ is investigated, for which a precise expression is obtained, valid in a certain region in the complex plane. The method of proof is based on complex integration and spectral theory of the non-Euclidean Laplacian.

1. INTRODUCTION

Laplace transforms play an important rôle in analytic number theory. Of special interest in the theory of the Riemann zeta-function $\zeta(s)$ are the Laplace transforms

$$L_k(s) := \int_0^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k} e^{-sx}\,\mathrm{d}x \qquad (k \in \mathbb{N},\ \Re\mathrm{e}\, s > 0). \tag{1.1}$$

E.C. Titchmarsh's well-known monograph [20, Chapter 7] gives a discussion of $L_k(s)$ when $s = \sigma$ is real and $\sigma \to 0+$, especially detailed in the cases $k = 1$ and $k = 2$. Indeed, a classical result of H. Kober [14] says that, as $\sigma \to 0+$,

$$L_1(2\sigma) = \frac{\gamma - \log(4\pi\sigma)}{2\sin\sigma} + \sum_{n=0}^{N} c_n\sigma^n + O(\sigma^{N+1}) \tag{1.2}$$

for any given integer $N \ge 1$, where the c_n's are effectively computable constants and $\gamma = 0.577\ldots$ is Euler's constant. For complex values of s the function $L_1(s)$

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was studied by F.V. Atkinson [1], and more recently by M. Jutila [13], who noted that Atkinson's argument gives

(1.3)

$$L_1(s) = -ie^{\frac{1}{2}is}(\log(2\pi) - \gamma + (\frac{\pi}{2} - s)i) + 2\pi e^{-\frac{1}{2}is}\sum_{n=1}^\infty d(n)\exp(-2\pi ine^{-is}) + \lambda_1(s)$$

in the strip $0 < \Re\mathrm{e}\, s < \pi$, where the function $\lambda_1(s)$ is holomorphic in the strip $|\Re\mathrm{e}\, s| < \pi$. Moreover, in any strip $|\Re\mathrm{e}\, s| \le \theta$ with $0 < \theta < \pi$, we have

$$\lambda_1(s) \ll_\theta (|s| + 1)^{-1}.$$

In [12] M. Jutila gave a discussion on the application of Laplace transforms to the evaluation of sums of coefficients of certain Dirichlet series.

F.V. Atkinson [2] obtained the asymptotic formula

$$L_2(\sigma) = \frac{1}{\sigma}(A\log^4\frac{1}{\sigma} + B\log^3\frac{1}{\sigma} + C\log^2\frac{1}{\sigma} + D\log\frac{1}{\sigma} + E) + \lambda_2(\sigma), \tag{1.4}$$

where $\sigma \to 0+$,

$$A = \frac{1}{2\pi^2},\ B = \pi^{-2}(2\log(2\pi) - 6\gamma + 24\zeta'(2)\pi^{-2})$$

and

$$\lambda_2(\sigma) \ll_\varepsilon \left(\frac{1}{\sigma}\right)^{\frac{13}{14}+\varepsilon}. \tag{1.5}$$

He also indicated how, by the use of estimates for Kloosterman sums, one can improve the exponent $\frac{13}{14}$ in (1.5) to $\frac{8}{9}$. This is of historical interest, since it is one of the first instances of application of Kloosterman sums to analytic number theory. Atkinson in fact showed that ($\sigma = \Re\mathrm{e}\, s > 0$)

$$L_2(s) = 4\pi e^{-\frac{1}{2}s}\sum_{n=1}^\infty d_4(n)K_0(4\pi i\sqrt{n}e^{-\frac{1}{2}s}) + \phi(s), \tag{1.6}$$

where $d_4(n)$ is the divisor function generated by $\zeta^4(s)$, K_0 is the Bessel function, and the series in (1.6) as well as $\phi(s)$ are both analytic in the region $|s| < \pi$. When $s = \sigma \to 0+$ one can use the asymptotic formula

$$K_0(z) = \tfrac{1}{2}\sqrt{\pi}z^{-1/2}e^{-z}\left(1 - 8z^{-1} + O(|z|^{-2})\right) \quad (|\arg z| < \theta < \frac{3\pi}{2},\ |z| \ge 1)$$

and then, by delicate analysis, one can deduce (1.4)–(1.5) from (1.6).

The author [5] gave explicit, albeit complicated expressions for the remaining coefficients C, D and E in (1.4). More importantly, he applied a result on the fourth moment of $|\zeta(\frac{1}{2}+it)|$, obtained jointly with Y. Motohashi [9], [11] (see also [4]), to establish that

$$\lambda_2(\sigma) \ll \sigma^{-1/2} \qquad (\sigma \to 0+), \tag{1.7}$$

and this is where the matter presently rests.

For $k \ge 3$ not much is known about $L_k(s)$, even when $s = \sigma \to 0+$. This is not surprising, since not much is known about upper bounds for

$$I_k(T) := \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \qquad (k \ge 3,\ k \in \mathbb{N}).$$

For a discsussion on $I_k(T)$ the reader is referred to the author's monographs [3] and [4]. One trivially has

$$I_k(T) \le e\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k} e^{-t/T}\,\mathrm{d}t = eL_k(\frac{1}{T}). \tag{1.8}$$

Thus any nontrivial bound of the form

$$L_k(\sigma) \ll_\varepsilon \left(\frac{1}{\sigma}\right)^{c_k+\varepsilon} \qquad (\sigma \to 0+,\ c_k \ge 1) \tag{1.9}$$

gives, in view of (1.8) ($\sigma = 1/T$), the bound

$$I_k(T) \ll_\varepsilon T^{c_k+\varepsilon}. \tag{1.10}$$

Conversely, if (1.10) holds, then we obtain (1.9) from the identity

$$L_k(\frac{1}{T}) = \frac{1}{T}\int_0^\infty I_k(t)e^{-t/T}\,\mathrm{d}t,$$

which is easily established by integration by parts.

2. SPECTRAL THEORY AND THE LAPLACE TRANSFORM of $|\zeta(\frac{1}{2}+ix)|^4$

The purpose of this paper is to consider $L_2(s)$, where s is a complex variable, and to prove a result analogous to (1.3), valid in a certain region in $\mathbb{C}$. We shall not use Atkinson's method and try to elaborate on (1.6). Our main tools are powerful methods from spectral theory, by which recently much advance has been made in connection with $I_2(T)$. For a competent and extensive account of spectral theory the reader is referred to Y. Motohashi's monograph [19]. Some of the relevant papers on $I_2(T)$ are [6]–[11], [14]–[18] and [21].

We begin by stating briefly the necessary notation involving the spectral theory of the non-Euclidean Laplacian. As usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ will denote the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})$–automorphic forms, and $\alpha_j = |\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue λ_j to which the Hecke series $H_j(s)$ is attached. We note that

$$\sum_{\kappa_j \le K} \alpha_j H_j^3(\tfrac{1}{2}) \ll K^2 \log^C K \qquad (C>0). \tag{2.1}$$

Our result is the following

THEOREM. *Let $0 \le \phi < \frac{\pi}{2}$ be given. Then for $0 < |s| \le 1$ and $|\arg s| \le \phi$ we have*

$$\begin{aligned} L_2(s) &= \frac{1}{s}\Big(A\log^4\frac{1}{s} + B\log^3\frac{1}{s} + C\log^2\frac{1}{s} + D\log\frac{1}{s} + E\Big) + G_2(s) \\ &+ s^{-\frac{1}{2}}\left\{\sum_{j=1}^{\infty}\alpha_j H_j^3(\tfrac{1}{2})\left(s^{-i\kappa_j}R(\kappa_j)\Gamma(\tfrac{1}{2}+i\kappa_j) + s^{i\kappa_j}R(-\kappa_j)\Gamma(\tfrac{1}{2}-i\kappa_j)\right)\right\}, \end{aligned} \tag{2.2}$$

where

$$R(y) := \sqrt{\frac{\pi}{2}}\left(2^{-iy}\frac{\Gamma(\frac{1}{4}-\frac{i}{2}y)}{\Gamma(\frac{1}{4}+\frac{i}{2}y)}\right)^3\Gamma(2iy)\cosh(\pi y) \tag{2.3}$$

and in the above region $G_2(s)$ is a regular function satisfying ($C>0$ is a suitable constant)

$$G_2(s) \ll |s|^{-1/2}\exp\left\{-\frac{C\log(|s|^{-1}+20)}{(\log\log(|s|^{-1}+20))^{2/3}(\log\log\log(|s|^{-1}+20))^{1/3}}\right\}. \tag{2.4}$$

Remark 1. The constants A, B, C, D, E in (2.2) are the same ones as in (1.4).

Remark 2. From Stirling's formula for the gamma-function it follows that $R(\kappa_j) \ll \kappa_j^{-1/2}$. In view of (2.1) this means that the series in (2.2) is absolutely convergent and uniformly bounded in s when $s=\sigma$ is real. Therefore, when $s=\sigma\to 0+$, (2.2) gives a refinement of (1.7).

Remark 3. From (1.4) and (1.7) it transpires that $\lambda(\sigma)$ is an error term when $0<\sigma<1$. For this reason we considered the values $0<|s|\le 1$ in (2.2), although one could treat the case $|s|>1$ as well.

Remark 4. From (2.2) and elementary properties of the Laplace transform one can easily obtain the Laplace transform of

$$E_2(T) := \int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t - TP_4(\log T), \quad P_4(x) = \sum_{j=0}^4 a_j x^j,$$

where $a_4 = 1/(2\pi^2)$ (for the evaluation of the remaining a_j's, see [5]).

3. PROOF OF THE THEOREM

Note first that in the integral defining $L_2(s)$ it suffices consider only the range of integration $[1,\infty]$, since the range $[0,1]$ trivially contributes $\ll 1$. We start from the well-known integral

$$e^{-z} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)z^{-s}\,\mathrm{d}s \qquad (\Re\mathrm{e}\, z>0, c>0), \tag{3.1}$$

and the function

$$\mathcal{Z}_2(w) := \int_1^\infty |\zeta(\tfrac{1}{2}+it)|^4 t^{-w}\,\mathrm{d}t \quad (\Re\mathrm{e}\, w>1),$$

introduced and studied by Y. Motohashi [17], [19]. Here as usual

$$\int_{(\sigma)} = \lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}.$$

Y. Motohashi showed that $\mathcal{Z}_2(s)$ has meromorphic continuation over $\mathbb{C}$. In the half-plane $\sigma = \Re\mathrm{e}\, s>0$ it has the following singularities: the pole $s=1$ of order 5, simple poles at $s=\frac{1}{2}\pm i\kappa_j$ ($\kappa_j = \sqrt{\lambda_j - \frac{1}{4}}$) and poles at $s=\frac{1}{2}\rho$, where ρ denotes complex zeros of $\zeta(s)$. The residue of $\mathcal{Z}_2(s)$ at $s=\frac{1}{2}+i\kappa_h$ equals

$$R_0(\kappa_h) = \sqrt{\frac{\pi}{2}}\Bigl(2^{-i\kappa_h}\frac{\Gamma(\frac{1}{4}-\frac{i}{2}\kappa_h)}{\Gamma(\frac{1}{4}+\frac{i}{2}\kappa_h)}\Bigr)^3\Gamma(2i\kappa_h)\cosh(\pi\kappa_h)\sum_{\kappa_j=\kappa_h}\alpha_j H_j^3(\tfrac{1}{2}),$$

and the residue at $s = \frac{1}{2} - i\kappa_h$ equals $\overline{R_0(\kappa_h)}$.

From (3.1) we have, for $c > 1$, $0 < |s| \le 1$ and $|\arg s| \le \phi$,

$$\begin{aligned}&\int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^4 e^{-sx}\,\mathrm{d}x\\ &= \int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^4 \left(\frac{1}{2\pi i}\int_{(c)} \Gamma(w)(sx)^{-w}\,\mathrm{d}w\right)\mathrm{d}x \qquad (3.2)\\ &= \frac{1}{2\pi i}\int_{(c)} \Gamma(w)s^{-w}\mathcal{Z}_2(w)\,\mathrm{d}w.\end{aligned}$$

In (3.2) we shift the line of integration to the contour $\mathcal{L}$ ($w = u + iv$) consisting of the curves

$$u = \tfrac{1}{2} - C\log^{-2/3}|v|(\log\log|v|)^{-1/3} \qquad (|v| \ge v_0 > 0,\ C > 0) \tag{3.3}$$

and the segment

$$u = u_0,\ u_0 = \tfrac{1}{2} - C\log^{-2/3}|v_0|(\log\log|v_0|)^{-1/3},\ |v| \le v_0.$$

Namely $\zeta(s) \ne 0$ (see e.g., [3, Chapter 6]) for

$$\sigma \ge 1 - A(\log t)^{-2/3}(\log\log t)^{-1/3} \qquad (s = \sigma + it,\ t \ge t_0 > 0,\ A > 0)$$

and a suitable constant A. The function $\mathcal{Z}_2(w)$ will be regular on $\mathcal{L}$, since the poles $w = \frac{1}{2}\rho$, $\zeta(\rho) = 0$) lie to the left of $\mathcal{L}$. For any given $\eta > 0$ one has

$$\mathcal{Z}_2(w) \ll e^{\eta|\Im\mathrm{m}\, w|} \qquad (w \in \mathcal{L}) \tag{3.4}$$

if C in (3.3) is taken sufficiently small. This follows easily from the proof of Lemma 1 of [7], which shows that the order of $\mathcal{Z}_2(w)$ is of the order given by (3.4) if $\Re\mathrm{e}\, w > 0$ and w stays away from the poles of $\mathcal{Z}_2(w)$. In the forthcoming work [8] it is even shown that in the above region $\mathcal{Z}_2(w)$ is of polynomial growth in $|v|$, which is more than what is required for our present purpose. Thus by the residue theorem we obtain from (3.2)

$$\int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^4 e^{-sx}\,\mathrm{d}x = \sum \operatorname{Res}\Gamma(w)s^{-w}\mathcal{Z}_2(w) + \frac{1}{2\pi i}\int_{\mathcal{L}} \Gamma(w)s^{-w}\mathcal{Z}_2(w)\,\mathrm{d}w, \tag{3.5}$$

and the last integral is regular for $0 < |s| \le 1$ and $|\arg s| \le \phi$. There are residues at $w = 1$ and at $w = \frac{1}{2} \pm i\kappa_j$. The contribution from $w = 1$ is (see (1.4))

$$\frac{1}{s}(A\log^4\frac{1}{s} + B\log^3\frac{1}{s} + C\log^2\frac{1}{s} + D\log\frac{1}{s} + E),$$

while the residues at $w = \frac12 \pm i\kappa_j$ yield

$$s^{-1/2}\left\{\sum_{j=1}^\infty \alpha_j H_j^3(\tfrac12)\left(s^{-i\kappa_j}R(\kappa_j)\Gamma(\tfrac12+i\kappa_j)+s^{i\kappa_j}R(-\kappa_j)\Gamma(\tfrac12-i\kappa_j)\right)\right\},$$

where $R(y)$ is defined by (2.3).

Write

$$\int_{\mathcal L}\Gamma(w)s^{-w}\mathcal Z_2(w)\,\mathrm dw \;=\; I_1+I_2,$$

say, where in I_1 we have $|v|\le V$, and in I_2 we have $|v|>V$, where $V(\gg 1)$ is a parameter to be chosen. Since

$$|s^{-w}| \;=\; |s|^{-u}e^{|v\arg s|}\quad(|\arg s|\le\phi),\;\; \Gamma(w)\ll |v|^{u-\frac12}e^{-\frac{\pi}{2}|v|}\quad(v\gg1),$$

then setting $\delta(x) = C\log^{-2/3}x(\log\log x)^{1/3}$ we obtain

$$I_1\ll |s|^{-u_0}+\int_{v_0}^V|s|^{-(\frac12-\delta(v))}e^{-(\frac{\pi}{2}-\phi)v}\,\mathrm dv\ll |s|^{-u_0}+|s|^{-(\frac12-\delta(V))}.$$

Similarly we have

$$I_2\ll |s|^{-\frac12}\int_V^\infty e^{-(\frac{\pi}{2}-\phi)v}\,\mathrm dv = |s|^{-\frac12}\,\frac{e^{-(\frac{\pi}{2}-\phi)V}}{(\frac{\pi}{2}-\phi)}.$$

Finally choosing

$$V \;=\; C_1\log(|s|^{-1}+20)$$

with suitable $C_1>0$ we obtain $(C_2>0)$

$$\begin{aligned}&\int_{\mathcal L}\Gamma(w)s^{-w}\mathcal Z_2(w)\,\mathrm dw\\ &\ll |s|^{-1/2}\exp\left\{-\frac{C_2\log(|s|^{-1}+20)}{(\log\log(|s|^{-1}+20))^{2/3}(\log\log\log(|s|^{-1}+20))^{1/3}}\right\},\end{aligned}$$

which in view of (3.5) completes the proof of the Theorem.

References

[1] F.V. Atkinson, The mean value of the zeta-function on the critical line, *Quart. J. Math. Oxford* **10**(1939), 122-128.

[2] F.V. Atkinson, The mean value of the zeta-function on the critical line, *Proc. London Math. Soc.* **47**(1941), 174-200.

[3] A. Ivić, The Riemann zeta-function, *John Wiley and Sons*, New York, 1985.

[4] A. Ivić, Mean values of the Riemann zeta-function, LN's **82**, *Tata Institute of Fundamental Research*, Bombay, 1991 (distr. by Springer Verlag, Berlin etc.).

[5] A. Ivić, On the fourth moment of the Riemann zeta-function, *Publs. Inst. Math. (Belgrade)* **57(71)**(1995), 101-110.

[6] A. Ivić, The Mellin transform and the Riemann zeta-function, in "*Proceedings of the Conference on Elementary and Analytic Number Theory (Vienna, July* 18-20, 1996)", Universität Wien & Universität für Bodenkultur, Eds. W.G. Nowak and J. Schoißengeier, Vienna 1996, 112-127.

[7] A. Ivić, On the error term for the fourth moment of the Riemann zeta-function, *J. London Math. Soc.* **60**(2)(1999), 21-32.

[8] A. Ivić, M. Jutila and Y. Motohashi, The Mellin transform of power moments of the zeta-function, *Acta Arithmetica* **95**(2000), 305-342.

[9] A. Ivić and Y. Motohashi, A note on the mean value of the zeta and L-functions VII, *Proc. Japan Acad. Ser. A* **66**(1990), 150-152.

[10] A. Ivić and Y. Motohashi, The mean square of the error term for the fourth moment of the zeta-function, *Proc. London Math. Soc.* (3)**66**(1994), 309-329.

[11] A. Ivić and Y. Motohashi, The fourth moment of the Riemann zeta-function, *J. Number Theory* **51**(1995), 16-45.

[12] M. Jutila, Mean values of Dirichlet series via Laplace transforms, in *"Analytic Number Theory"* (ed. Y. Motohashi), London Math. Soc. LNS **247**, *Cambridge University Press*, Cambridge, 1997, 169-207.

[13] M. Jutila, Atkinson's formula revisited, in *"Voronoi's Impact on Modern Science"*, Book 1, Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, 1998, 139-154.

[14] H. Kober, Eine Mittelwertformel der Riemannschen Zetafunktion, *Compositio Math.* **3**(1936), 174-189.

[15] Y. Motohashi, The fourth power mean of the Riemann zeta-function, in *"Proceedings of the Amalfi Conference on Analytic Number Theory 1989"*, eds. E. Bombieri et al., Università di Salerno, Salerno, 1992, 325-344.

[16] Y. Motohashi, An explicit formula for the fourth power mean of the Riemann zeta-function, *Acta Math.* **170**(1993), 181-220.

[17] Y. Motohashi, A relation between the Riemann zeta-function and the hyperbolic Laplacian, *Annali Scuola Norm. Sup. Pisa, Cl. Sci. IV ser.* **22**(1995), 299-313.

[18] Y. Motohashi, The Riemann zeta-function and the non-Euclidean Laplacian, *Sugaku Expositions*, AMS **8**(1995), 59-87.

[19] Y. Motohashi, Spectral theory of the Riemann zeta-function, *Cambridge University Press*, Cambridge, 1997.

[20] E.C. Titchmarsh, The theory of the Riemann zeta-function (2nd ed.), Clarendon Press, Oxford, 1986.

[21] N.I. Zavorotnyi, On the fourth moment of the Riemann zeta-function (in Russian), in *"Automorphic functions and number theory I"*, Coll. Sci. Works, Vladivostok, 1989, 69-125.

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